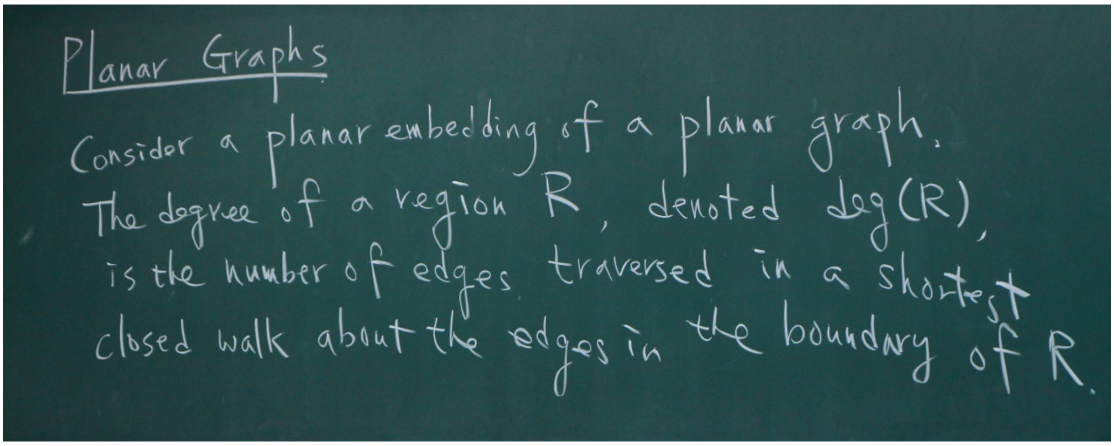
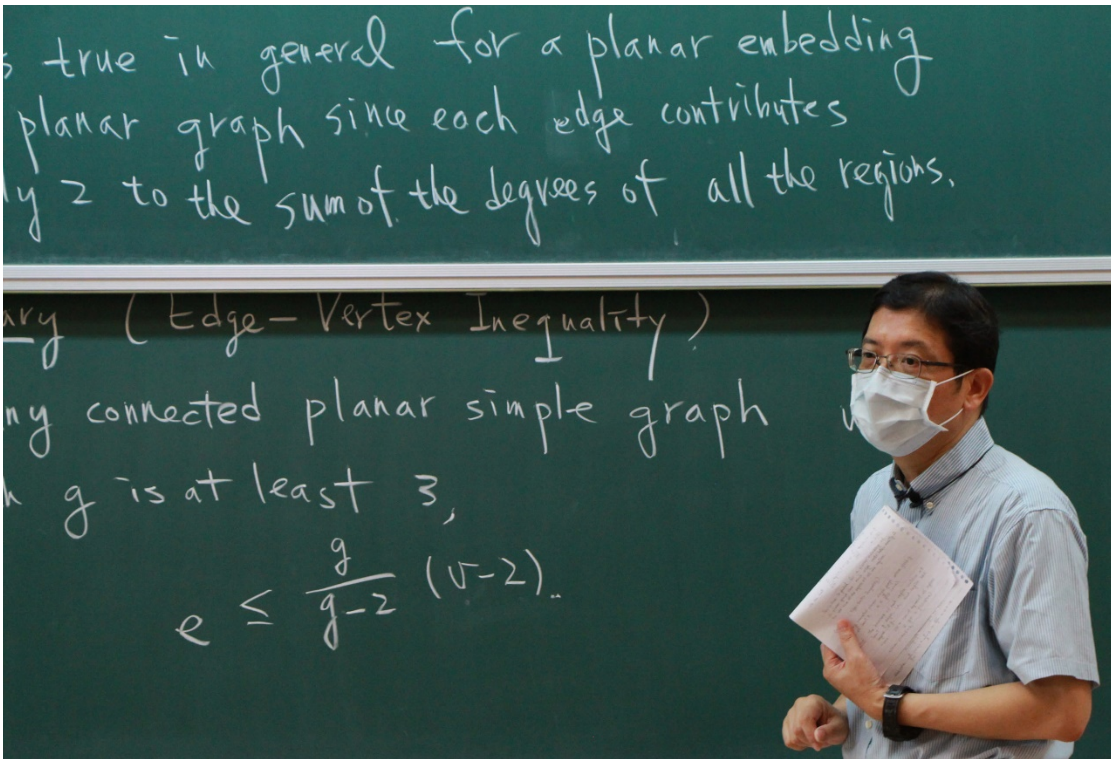
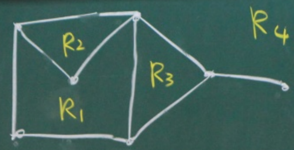
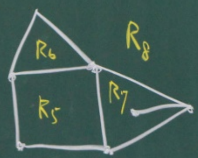
【10920 趙啟超教授離散數學 / 第 22 堂版書】





Example





$\deg(R_1) = 5$

$\deg(R_2) = 3$

$\deg(R_3) = 3$

$\deg(R_4) = 7$

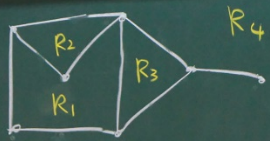
$\deg(R_5) = 4$ $\quad \deg(R_7) = 5$

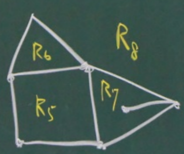
$\deg(R_6) = 3$ $\quad \deg(R_8) = 6$

Planar Graphs

Consider a planar embedding of a planar graph.
The degree of a region R, denoted $\deg(R)$,
is the number of edges traversed in a shortest
closed walk about the edges in the boundary of R.

Example





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We see that $\sum_{i=1}^{4} \deg(R_i) = 18 = 2\cdot|E|$

and $\sum_{i=5}^{8} \deg(R_i) = 18 = 2\cdot|E|$

This is true in general for a planar embedding of a planar graph since each edge contributes exactly 2 to the sum of the degrees of all the regions.

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Corollary (Edge-Vertex Inequality)

In any connected planar simple graph whose girth g is at least 3,

$$e \le \frac{g}{g-2}(v-2).$$

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<u>Corollary</u> (Edge-Vertex Inequality)

In any connected planar simple graph whose girth g is at least 3,

$$e \leq \frac{g}{g-2}(v-2).$$

<u>Proof</u> The boundary of each region contains at least g edges, i.e., each region has degree $\geq g$.

Hence $2e =$ sum of degrees of all regions

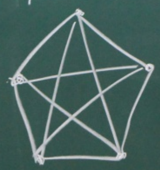
$\geq gr = g(2-v+e)$

since $v-e+r=2$.

Therefore, $(g-2)e \le g(v-2)$

which yields $e \le \dfrac{g}{g-2}(v-2)$.

<u>Example</u>



K_5

$v = 5$

$e = 10$

$g = 3$

$\dfrac{g}{g-2}(v-2) = \dfrac{3}{3-2}(5-2)$

$= \dfrac{3}{1} \cdot 3 = 9 < 10 = e.$

$\therefore K_5$ is nonplanar.

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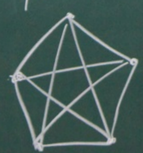

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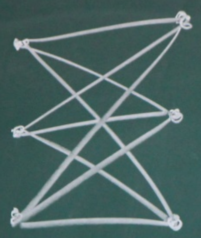


$K_{3,3}$

$v = 6$

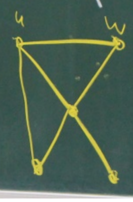
$e = 9$

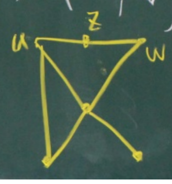
$g = 4$

$$\frac{g}{g-2}(v-2) = \frac{4}{2}\cdot 4 = 8 < 9 = e$$

$\therefore$ $K_{3,3}$ is nonplanar.

Def Let $G=(V,E)$ be an undirected simple graph, with $E \neq \phi$. An elementary subdivision of G results when an edge $\{u,w\}$ is removed from G and then the edges $\{u,z\}$, $\{z,w\}$ are added, where $z \notin V$.





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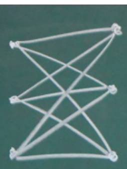

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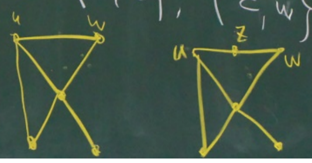

Def The undirected simple graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ are called homeomorphic if they are isomorphic or they can both be obtained from the same undirected simple graph H by a sequence of elementary subdivisions.

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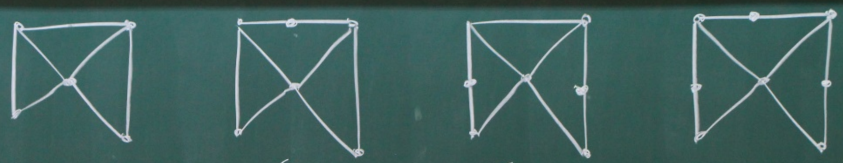

G_1 G_2 G_3 G_4

G_1, G_2, G_3, and G_4 are homeomorphic.

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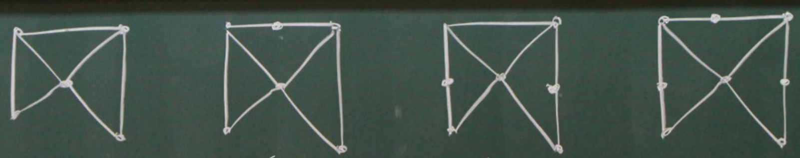

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Def A graph $G_1=(V_1,E_1)$ is called a subgraph of a graph $G=(V,E)$ if $V_1 \subseteq V$ and $E_1 \subseteq E$.

Theorem (Kuratowski's Theorem)
A graph is nonplanar iff it contains a subgraph that is homeomorphic to either K_5 or $K_{3,3}$.

Proof If a graph G has a subgraph homeomorphic to either K_5 or $K_{3,3}$, it is clear that G is nonplanar. The converse of this theorem, however, is much more complicated to prove, and the proof will not be given here. ∎

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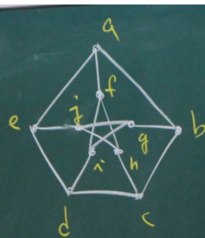

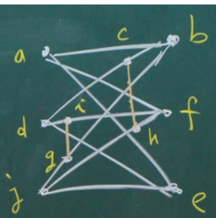

$\therefore$ Petersen graph is nonplanar (since it contains a subgraph homeomorphic to $K_{3,3}$).

Also note $\dfrac{g}{g-2}(v-2) = \dfrac{5}{5-2}(10-2)$

$$= \frac{5}{3} \cdot 8 = \frac{40}{3} < e = 15.$$

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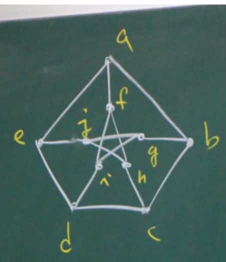

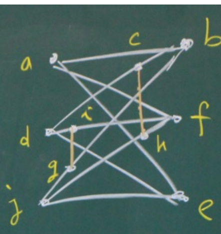

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